Project # 1

1. Prove that the metric space $(C([0,1]), d)$ is complete, where the metric d is defined as

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

2. Prove that the metric space $(C([0,1]), d)$ is not complete, where the metric d is defined as

$$d(f,g) = \int_0^1 |f(x) - g(x)| \,\mathrm{d}x.$$

Hint: Consider the follownig sequence $\{f_n \in C([0,1])\colon n \in \mathbb{N}_{\geq 1}\}$

$$f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{3}] \\ \text{linear} & x \in [\frac{1}{3}, \frac{1}{3} + \frac{2}{3}\cdot\frac{1}{2^n}] \\ 1 & x \in [\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{2^n}, 1] \end{cases}$$

Solution:

1.

Proof. For any Cauchy sequence $\{f_n\colon n \in \mathbb{N}_{\geq 1}\}$, we will show that it converges to certain $f \in C[0,1]$.

Assume that $\{f_n \colon n \in \mathbb{N}_{\geq 1}\}$ is Cauchy, then for any $x \in [0,1]$, according to the definition of metric d on $C[0,1]$, we know that $\{f_n(x) \colon n \in \mathbb{N}_{\geq 1}\}$ is a Cauchy sequence in $\mathbb{C}$. As $\mathbb{C}$ is complete, it must converge in $\mathbb{C}$. Define a function $f \colon [0,1] \longrightarrow \mathbb{C}$, such that $f(x)$ is chosen to be the limit of $\{f_n(x) \colon n \in \mathbb{N}_{\geq 1}\}$ for all $x \in [0,1]$.

It only remains to show that f is continuous on $[0,1]$ (that is, $f \in C[0,1]$) and $d(f_n, f) \to 0$ as $n \to \infty$.

We will first prove the following claim.

Claim: For any $\epsilon > 0$, there exists $N \in \mathbb{N}_{\geq 1}$, such that for any $n > N$, we have

$$\sup_{x \in [0,1]} |f(x) - f_n(x)| < \epsilon.$$

Proof of this claim: As $\{f_n \colon n \in \mathbb{N}_{\geq 1}\}$ is a Cauchy sequence, for any $\epsilon > 0$, there exists $N \in \mathbb{N}_{\geq 1}$, such that for all $n > N, k \in \mathbb{N}$, we have

$$\sup_{x \in [0,1]} |f_n(x) - f_{n+k}(x)| < \epsilon/4.$$

As $\lim_{k\to\infty} f_{n+k}(x) = f(x)$ for all $x \in X$, we have that

$$|f_n(x) - f(x)| \leq \epsilon/4 < \epsilon/2$$

for all $x \in X$ and for all $n > N$, which finishes the proof of the claim.

Based on this claim, we can easily finish our proof. In fact, we just need to show that $f \in C[0,1]$, as $d(f_n, f) \to 0$ then follows directly from our claim.

For any $a \in [0,1]$, we will show that f is continuous at a. In fact, according to the claim, we can find $N \in \mathbb{N}_{\geq 1}$ such that

$$\sup_{x \in [0,1]} |f_N(x) - f(x)| < \epsilon/3.$$

As f_N is continuous, there exists $\delta > 0$, such that for any $x \in [0,1]$ with $|x - a| < \delta$, we have

$$|f_N(x) - f_N(a)| < \epsilon/3.$$

Then we have, for any $x \in [0,1]$ with $|x-a| < \delta$,

$$\begin{aligned}
|f(x) - f(a)| &= |f(x) - f_n(x)) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\
&\leq \sup_{x\in[0,1]} |f_N(x) - f(x)| + |f_n(x) - f_n(a)| + \sup_{x\in[0,1]} |f_N(x) - f(x)| \\
&\leq \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon.
\end{aligned}$$

Thus f is continuous at any $a \in [0,1]$, which finishes the proof.

$\square$

2. Not so hard to prove. Need the fact that for $\{f_n \in C[0,1]\}_{n\in\mathbb{N}_{\geq 1}}$ and $f \in C[0,1]$, if $\int_0^1 |f_n(x) - f(x)| \, \mathrm{d}x \to 0$, then $f_n(x) \to f(x)$ for all $x \in [0,1]$. Note that the proof of this fact is one of the homework problem already assigned to the class.